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On a Θ-Function Formula.

By Thomas Craig, *Johns Hopkins University.*

The notation adopted in the following is that used by Clifford in his paper "Algebraic Introduction to Elliptic Functions," contained in the volume of his "Mathematical Papers." I may just give the definitions of the four different Θ-functions in this notation: they are

$$\begin{aligned}\Theta(u, a) &= \Sigma e^{n^2 a + 2nu},\\ \Theta'(u, a) &= \Sigma^{(-)n} e^{n^2 a + 2nu},\\ \Theta_1(u, a) &= \Sigma e^{(n+\frac{1}{2})^2 a + 2(n+\frac{1}{2})u}.\\ \Theta'_1(u, a) &= \Sigma^{(-)n} e^{(n+\frac{1}{2})^2 a + 2(n+\frac{1}{2})u}\end{aligned}$$

The relations connecting these are of course

$$\Theta\left(u + \frac{\pi i}{2}, a\right) = \Theta'(u, a) \qquad \Theta'\left(u + \frac{\pi i}{2}, a\right) = \Theta(u, a)$$

$$\Theta\left(u + \frac{a}{2}, a\right) = e^{-n-\frac{a}{4}}\Theta_1(u, a) \qquad \Theta_1\left(u + \frac{a}{2}, a\right) = e^{-n-\frac{a}{4}}\Theta(u, a)$$

$$\Theta\left(u + \frac{\pi i}{2} + \frac{a}{2}, a\right) = e^{-n-\frac{a}{4}}\Theta'_1(u, a) \quad \Theta'_1\left(u + \frac{\pi i}{2} + \frac{a}{2}, a\right) = ie^{-n-\frac{a}{4}}\Theta(u, a).$$

The quantities $\frac{\pi i}{2}$, $\frac{a}{2}$, $\frac{\pi i}{2} + \frac{a}{2}$ may be conveniently spoken of as the first, second and third quadrants.

On page 448 of the "Mathematical Papers" occurs the well-known formula

$$\begin{aligned}\Theta(u)\dot{\Theta}(v) - \dot{\Theta}(u)\Theta(v) = {}& 2\Theta'(v - u, 2a)\Theta(v + u, 2a)\\ &+ 2\dot{\Theta}_1(v - u, 2a)\Theta_1(v + u, 2a)\end{aligned}$$

(whenever the modulus is not written it will be understood to be a, when it has any other value it will always be written down). From this follow, by giving special values to u and v, certain well-known and important formulae. The form of the left-hand number of this equation suggests writing it as

$$\begin{vmatrix}\Theta(u), & \dot{\Theta}(u)\\ \Theta(v) & \dot{\Theta}(v)\end{vmatrix} = 2\dot{\Theta}(v - u, 2a)\dot{\Theta}(v + u, 2a) + 2\dot{\Theta}_1(v - u, 2a)\Theta_1(v + u, 2a).$$

In what follows I examine the determinant

$$\begin{vmatrix} \Theta(u), & \dot\Theta(u), & \ddot\Theta(u), & \dddot\Theta(u) \\ \Theta(v), & \dot\Theta(v), & \ddot\Theta(v), & \dddot\Theta(v) \\ \Theta(w), & \dot\Theta(w), & \ddot\Theta(w), & \dddot\Theta(w) \\ \Theta(t), & \dot\Theta(t), & \ddot\Theta(t), & \dddot\Theta(t) \end{vmatrix}, \text{ say } \Delta.$$

The dots of course denote differentiation with respect to the argument.

We have

$$\Theta(u) = \sum_{\alpha=-\infty}^{\alpha=\infty} e^{\alpha^2 a + 2\alpha u},$$

$$\Theta(v) = \sum_{\beta=-\infty}^{\beta=\infty} e^{\beta^2 a + 2\beta v},$$

$$\Theta(w) = \sum_{\gamma=-\infty}^{\gamma=\infty} e^{\gamma^2 a + 2\gamma w},$$

$$\Theta(t) = \sum_{\delta=-\infty}^{\delta=\infty} e^{\delta^2 a + 2\delta t},$$

It is unnecessary to write down hereafter the limits of the summation which is always understood to extend from $-\infty$ to $+\infty$. On substituting the values of $\Theta(u)$, $\Theta(v)$, $\Theta(w)$, $\Theta(t)$ in the above determinant and making some easy reductions it becomes

$$\begin{aligned} .\Delta = 64\sum_\alpha\sum_\beta\sum_\gamma\sum_\delta \{\alpha^2\beta^2 + \gamma^2\delta^2)(\beta-\alpha)(\delta-\gamma) + (\beta^2\delta^2 + \alpha^2\gamma^2)(\gamma-\alpha)(\beta-\delta) \\ + (\beta^2\gamma^2 + \delta^2\alpha^2)(\delta-\alpha)(\gamma-\beta)\} \text{ exp. } [(\alpha^2+\beta^2+\gamma^2+\delta^2)\, a \\ + 2\alpha u + 2\beta v + 2\gamma w + 2\delta t] \end{aligned}$$

where exp. $[H]$ stands for e^H. Write this as

$$\Delta = \sum_\alpha\sum_\beta\sum_\gamma\sum_\delta 64\,(\alpha, \beta, \gamma, \delta) \text{ exp. } [H].$$

It is very easy to see that

$$(\alpha, \beta, \gamma, \delta) = \begin{vmatrix} 1, & 1, & 1, & 1 \\ \alpha, & \beta, & \gamma, & \delta \\ \alpha^2, & \beta^2, & \gamma^2, & \delta^2 \\ \alpha^3, & \beta^3, & \gamma^3, & \delta^3 \end{vmatrix}$$

or, using the ordinary notation for this,

$$(\alpha, \beta, \gamma, \delta) = \zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta),$$

and we know that

$$\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta) = (\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta).$$

Introduce four new quantities α_1, β_1, γ_1, δ_1, defined by the equations

$$\begin{aligned}
\alpha_1 &= \alpha-\beta-\gamma-\delta,\\
\beta_1 &= -\alpha+\beta-\gamma-\delta,\\
\gamma_1 &= -\alpha-\beta+\gamma-\delta,\\
\delta_1 &= -\alpha-\beta-\gamma+\delta;
\end{aligned}$$

from these we have the relations

$$\alpha^2+\beta^2+\gamma^2+\delta^2 = \tfrac{1}{4}(\alpha_1^2+\beta_1^2+\gamma_1^2+\delta_1^2)$$

and

$$\begin{aligned}
\alpha &= \tfrac{1}{4}(\alpha_1-\beta_1-\gamma_1-\delta_1),\\
\beta &= \tfrac{1}{4}(-\alpha_1+\beta_1-\gamma_1-\delta_1),\\
\gamma &= \tfrac{1}{4}(-\alpha_1-\beta_1+\gamma_1-\delta_1),\\
\delta &= \tfrac{1}{4}(-\alpha_1-\beta_1-\gamma_1+\delta_1).
\end{aligned}$$

It is clear by inspection of the above values for α_1, β_1, γ_1, δ_1, that whatever (integral) values α, β, γ, δ may have these are either all even or all odd, and so there are only two cases to examine, viz.

$$\begin{aligned}
&\text{I.} \quad \alpha_1, \beta_1, \gamma_1, \delta_1 = 2f,\ 2g,\ 2h,\ 2k.\\
&\text{II.} \quad \alpha_1, \beta_1, \gamma_1, \delta_1 = 2f+1,\ 2g+1,\ 2h+1,\ 2k+1.
\end{aligned}$$

The exponent H expressed in terms of the new quantities α_1, etc., is

$$\begin{aligned}
H = {}& \tfrac{1}{2}\alpha_1(u-v-w-t) + \tfrac{1}{2}\beta_1(-u+v-w-t)\\
&+ \tfrac{1}{2}\gamma_1(-u-v+w-t) + \tfrac{1}{2}\delta_1(-u-v-w+t);
\end{aligned}$$

also the factor which multiplies exp. $[H]$, viz.

$$64\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta)$$

becomes

$$\zeta^{\frac{1}{2}}(\alpha_1, \beta_1, \gamma_1, \delta_1).$$

Substituting these values in the expression for Δ this becomes

$$\begin{aligned}
\Delta = \sum\sum\sum_{\alpha_1\,\beta_1\,\gamma_1\,\delta_1}\sum\zeta^{\frac{1}{2}}(\alpha_1, \beta_1, \gamma_1, \delta_1) \text{ exp. } &\left[\tfrac{1}{4}(\alpha_1^2+\beta_1^2+\gamma_1^2+\delta_1^2) + \tfrac{1}{2}\alpha_1(u-v-w-t)\right.\\
&\left.+ \tfrac{1}{2}\beta_1(-u+v-w-t) + \tfrac{1}{2}\gamma_1(-u-v+w-t) + \tfrac{1}{2}\delta_1(-u-v-w+t)\right]
\end{aligned}$$

where

$$\zeta^{\frac{1}{2}}(\alpha_1, \beta_1, \gamma_1, \delta_1) = (\alpha_1-\beta_1)(\alpha_1-\gamma_1)(\alpha_1-\delta_1)(\beta_1-\gamma_1)(\beta_1-\delta_1)(\gamma_1-\delta_1).$$

There are two cases to be examined, viz. case I. of the above where α_1, β_1, γ_1, δ_1 are all even, and case II. where they are all odd. The results obtained for the two separate cases will give on being added together the complete value of Δ.

In analogy with the relations connecting α, β, etc. with α_1, β_1, etc., it will be convenient to write

$$\begin{aligned} u_1 &= \quad u - v - w - t \\ v_1 &= -u + v - w - t \\ w_1 &= -u - v + w - t \\ t_1 &= -u - v - w + t \end{aligned}$$

from which follow

$$u = \tfrac{1}{4}(u_1 - v_1 - w_1 - t_1) \text{ etc.}$$

$$\Sigma u^2 = \tfrac{1}{4}\Sigma u_1^2.$$

Now we have

$$\Delta = \sum_{\alpha_1}\sum_{\beta_1}\sum_{\gamma_1}\sum_{\delta_1}\zeta^{\frac{1}{2}}(\alpha_1, \beta_1, \gamma_1, \delta_1)\ \text{exp.}\ \left[\tfrac{1}{4}(\alpha_1^2 + \beta_1^2 + \gamma_1^2 + \delta_1^2)a \right.$$
$$\left. + \tfrac{1}{2}\alpha_1 u_1 + \tfrac{1}{2}\beta_1 v_1 + \tfrac{1}{2}\gamma_1 w_1 + \tfrac{1}{2}\delta_1 t_1\right].$$

Give now the even values to α_1, β_1, γ_1, δ_1, and we have clearly

$$\zeta^{\frac{1}{2}}(\alpha_1, \beta_1, \gamma_1, \delta_1) = 64\zeta^{\frac{1}{2}}(f, g, h, k);$$

further the exponential factor becomes

$$\text{exp.}\ [(f^2 + g^2 + h^2 + k^2)a + fu_1 + gv_1 + hw_1 + kt_1].$$

Combining these we have

$$\Delta_1 = \sum_{f}\sum_{g}\sum_{h}\sum_{k} 64\zeta^{\frac{1}{2}}(f, g, h, k)\ \text{exp.}\ [(f^2 + g^2 + h^2 + k^2)a$$
$$+ fu_1 + gv_1 + hw_1 + kt_1]$$

or

$$\Delta_1 = \begin{vmatrix} \Theta\left(\frac{u_1}{2}\right) & \dot{\Theta}\left(\frac{u_1}{2}\right) & \ddot{\Theta}\left(\frac{u_1}{2}\right) & \dddot{\Theta}\left(\frac{u_1}{2}\right) \\ \Theta\left(\frac{v_1}{2}\right) & \dot{\Theta}\left(\frac{v_1}{2}\right) & \ddot{\Theta}\left(\frac{v_1}{2}\right) & \dddot{\Theta}\left(\frac{v_1}{2}\right) \\ \Theta\left(\frac{w_1}{2}\right) & \dot{\Theta}\left(\frac{w_1}{2}\right) & \ddot{\Theta}\left(\frac{w_1}{2}\right) & \dddot{\Theta}\left(\frac{w_1}{2}\right) \\ \Theta\left(\frac{t_1}{2}\right) & \dot{\Theta}\left(\frac{t_1}{2}\right) & \ddot{\Theta}\left(\frac{t_1}{2}\right) & \dddot{\Theta}\left(\frac{t_1}{2}\right) \end{vmatrix}.$$

Again take the odd values of α_1, β_1, γ_1, δ_1, these give

$$\zeta^{\frac{1}{2}}(\alpha_1, \beta_1, \gamma_1, \delta_1) = 64\zeta^{\frac{1}{2}}(f, g, h, k).$$

The exponential factor now is

$$\text{exp.}\ \left[\left((f + \tfrac{1}{2})^2 + (g + \tfrac{1}{2})^2 + (h + \tfrac{1}{2})^2 + (k + \tfrac{1}{2})^2\right)a\right.$$
$$\left. + (f + \tfrac{1}{2})u_1 + (g + \tfrac{1}{2})v_1 + (h + \tfrac{1}{2})w_1 + (k + \tfrac{1}{2})t_1\right].$$

Combining these we have

$$\Delta_2 = \sum_{f}\sum_{g}\sum_{h}\sum_{k} 64\zeta^{\frac{1}{2}}(f, g, h, k)\ \text{exp.}\ \left[\left((f + \tfrac{1}{2})^2 + (g + \tfrac{1}{2})^2 + (h + \tfrac{1}{2})^2 + (k + \tfrac{1}{2})^2\right)a\right.$$
$$\left. + (f + \tfrac{1}{2})u_1 + (g + \tfrac{1}{2})v_1 + (h + \tfrac{1}{2})w_1 + (k + \tfrac{1}{2})t_1\right],$$

and since

$$\zeta^{\frac{1}{2}}(f+\tfrac{1}{2},\, g+\tfrac{1}{2},\, h+\tfrac{1}{2},\, k+\tfrac{1}{2}) = \zeta^{\frac{1}{2}}(f,\, g,\, h,\, k)$$

this becomes

$$\Delta_2 = \begin{vmatrix} \Theta_1\left(\frac{u_1}{2}\right), & \dot{\Theta}_1\left(\frac{u_1}{2}\right), & \ddot{\Theta}_1\left(\frac{u_1}{2}\right), & \dddot{\Theta}_1\left(\frac{u_1}{2}\right) \\ \Theta_1\left(\frac{v_1}{2}\right), & \dot{\Theta}_1\left(\frac{v_1}{2}\right), & \ddot{\Theta}_1\left(\frac{v_1}{2}\right), & \dddot{\Theta}_1\left(\frac{v_1}{2}\right) \\ \Theta_1\left(\frac{w_1}{2}\right), & \dot{\Theta}_1\left(\frac{w_1}{2}\right), & \ddot{\Theta}_1\left(\frac{w_1}{1}\right), & \dddot{\Theta}_1\left(\frac{w_1}{2}\right) \\ \Theta_1\left(\frac{t_1}{2}\right), & \dot{\Theta}_1\left(\frac{t_1}{2}\right), & \ddot{\Theta}_1\left(\frac{t_1}{2}\right), & \dddot{\Theta}_1\left(\frac{t_1}{1}\right) \end{vmatrix}$$

Now $\Delta = \Delta_1 + \Delta_2$, and so we have the final formula

$$\begin{vmatrix} \Theta(u) & \dot{\Theta}(u) & \ddot{\Theta}(u) & \dddot{\Theta}(u) \\ \Theta(v) & \dot{\Theta}(v) & \ddot{\Theta}(v) & \dddot{\Theta}(v) \\ \Theta(w) & \dot{\Theta}(w) & \ddot{\Theta}(w) & \dddot{\Theta}(w) \\ \Theta(t) & \dot{\Theta}(t) & \ddot{\Theta}(t) & \dddot{\Theta}(t) \end{vmatrix} = \begin{vmatrix} \Theta\left(\frac{u_1}{2}\right) & \dot{\Theta}\left(\frac{u_1}{2}\right) & \ddot{\Theta}\left(\frac{u_1}{2}\right) & \dddot{\Theta}\left(\frac{u_1}{2}\right) \\ \Theta\left(\frac{v_1}{2}\right) & \dot{\Theta}\left(\frac{v_1}{2}\right) & \ddot{\Theta}\left(\frac{v_1}{2}\right) & \dddot{\Theta}\left(\frac{v_1}{2}\right) \\ \Theta\left(\frac{w_1}{2}\right) & \dot{\Theta}\left(\frac{w_1}{2}\right) & \ddot{\Theta}\left(\frac{w_1}{2}\right) & \dddot{\Theta}\left(\frac{w_1}{2}\right) \\ \Theta\left(\frac{t_1}{2}\right) & \dot{\Theta}\left(\frac{t_1}{2}\right) & \ddot{\Theta}\left(\frac{t_1}{2}\right) & \dddot{\Theta}\left(\frac{t_1}{2}\right) \end{vmatrix}$$

$$+ \begin{vmatrix} \Theta_1\left(\frac{u_1}{2}\right) & \dot{\Theta}_1\left(\frac{u_1}{2}\right) & \ddot{\Theta}_1\left(\frac{u_1}{2}\right) & \dddot{\Theta}_1\left(\frac{u_1}{2}\right) \\ \Theta_1\left(\frac{v_1}{2}\right) & \dot{\Theta}_1\left(\frac{v_1}{2}\right) & \ddot{\Theta}_1\left(\frac{v_1}{2}\right) & \dddot{\Theta}_1\left(\frac{v_1}{2}\right) \\ \Theta_1\left(\frac{w_1}{2}\right) & \dot{\Theta}_1\left(\frac{w_1}{2}\right) & \ddot{\Theta}_1\left(\frac{w_1}{2}\right) & \dddot{\Theta}_1\left(\frac{w_1}{2}\right) \\ \Theta_1\left(\frac{t_1}{2}\right) & \dot{\Theta}_1\left(\frac{t_1}{2}\right) & \ddot{\Theta}_1\left(\frac{t_1}{2}\right) & \dddot{\Theta}_1\left(\frac{t_1}{2}\right) \end{vmatrix}$$

The functions Θ and Θ_1 are even functions, so in this equation we might replace $\frac{u_1}{2}$, $\frac{v_1}{2}$, $\frac{w_1}{2}$, $\frac{t_1}{2}$ by $-\frac{u_1}{2}$, $-\frac{v_1}{2}$, $-\frac{w_1}{2}$, $-\frac{t_1}{2}$ without altering the values of either Θ or Θ_1. Assume now the particular values

$$u = u,$$
$$v = u + \frac{\pi i}{2},$$
$$w = u + \frac{a}{2},$$
$$t = u + \frac{\pi i}{2} + \frac{a}{2}.$$

We find now

$$\frac{u_1}{2} = -u - \frac{\pi i}{2} - \frac{a}{2},$$

$$\frac{v_1}{2} = -u - \frac{a}{2},$$

$$\frac{w_1}{2} = -u - \frac{\pi i}{2},$$

$$\frac{t_1}{2} = -u \cdot$$

Substitute these values in the above formula, observing the known relations

$$\Theta\left(u + \frac{\pi i}{2}\right) = \Theta'(u) \qquad \Theta_1\left(u + \frac{\pi i}{2}\right) = i\Theta'_1(u)$$

$$\Theta\left(u + \frac{a}{2}\right) e^{-n-\frac{a}{4}}\Theta_1(u) \qquad \Theta_1\left(u + \frac{a}{2}\right) = e^{-n-\frac{a}{4}}\Theta(u)$$

$$\Theta\left(u + \frac{\pi i}{2} + \frac{a}{2}\right) = e^{-n-\frac{a}{4}}\Theta'_1(u) \qquad \Theta_1\left(u + \frac{\pi i}{2} + \frac{a}{2}\right) = -ie^{-n-\frac{a}{4}}\Theta'(u)$$

and we find without difficulty

$$\begin{vmatrix} \Theta, & \dot{\Theta} & \ddot{\Theta} & \dddot{\Theta} \\ \Theta', & \dot{\Theta}' & \ddot{\Theta}' & \dddot{\Theta}' \\ \Theta_1, & \dot{\Theta}_1 - \Theta_1, & \ddot{\Theta}_1 - 2\dot{\Theta}_1 + \Theta_1, & \dddot{\Theta}_1 - 3\ddot{\Theta}_1 + 3\dot{\Theta}_1 - \Theta_1 \\ \Theta'_1, & \dot{\Theta}'_1 - \Theta'_1, & \ddot{\Theta}'_1 - 2\dot{\Theta}'_1 + \Theta'_1, & \dddot{\Theta}'_1 - 3\ddot{\Theta}'_1 + 3\dot{\Theta}'_1 - \Theta'_1 \end{vmatrix}$$

$$= \begin{vmatrix} \Theta'_1, & \dot{\Theta}'_1 - \Theta'_1, & \ddot{\Theta}'_1 - 2\dot{\Theta}'_1 + \Theta'_1, & \dddot{\Theta}'_1 - 3\ddot{\Theta}'_1 + 3\dot{\Theta}'_1 - \Theta'_1 \\ \Theta_1, & \dot{\Theta}_1 - \Theta_1, & \ddot{\Theta}_1 - 2\dot{\Theta}_1 + \Theta_1, & \dddot{\Theta}_1 - 3\ddot{\Theta}_1 + 3\dot{\Theta}_1 - \Theta_1 \\ \Theta', & \dot{\Theta}', & \ddot{\Theta}', & \dddot{\Theta}' \\ \Theta, & \dot{\Theta}, & \ddot{\Theta}, & \dddot{\Theta} \end{vmatrix}$$

$$+ \begin{vmatrix} \Theta', & \dot{\Theta}' - \Theta', & \ddot{\Theta}' - 2\dot{\Theta}' + \Theta', & \dddot{\Theta}' - 3\ddot{\Theta}' + 3\dot{\Theta}' - \Theta' \\ \Theta, & \dot{\Theta} - \Theta, & \ddot{\Theta} - 2\dot{\Theta} + \Theta, & \dddot{\Theta} - 3\ddot{\Theta} + 3\dot{\Theta} - \Theta \\ \Theta'_1, & \dot{\Theta}'_1, & \ddot{\Theta}'_1, & \dddot{\Theta}'_1 \\ \Theta_1, & \dot{\Theta}_1, & \ddot{\Theta}_1, & \dddot{\Theta}_1 \end{vmatrix}$$

A common factor $e^{-2n-\frac{a}{2}}$ has been divided out of both sides of this equation.

Arranging the rows in the first determinant on the right-hand side of this equation so that they shall be in the same order as those of the determinant on the left, we see that this determinant is identical with that on the left-hand side, and so we arrive at the remarkable result

$$\begin{vmatrix} \Theta', & \dot{\Theta}' - \Theta', & \ddot{\Theta}' - 2\dot{\Theta}' + \Theta', & \dddot{\Theta}' - 3\ddot{\Theta}' + 3\dot{\Theta}' - \Theta' \\ \Theta, & \dot{\Theta} - \dot{\Theta}, & \ddot{\Theta} - 2\dot{\Theta} + \Theta, & \dddot{\Theta} - 3\ddot{\Theta} + 3\dot{\Theta} - \Theta \\ \Theta'_1, & \dot{\Theta}'_1, & \ddot{\Theta}'_1, & \dddot{\Theta}'_1, \\ \Theta_1, & \dot{\Theta}_1, & \ddot{\Theta}_1, & \dddot{\Theta}_1, \end{vmatrix} = 0$$

Interchange the first and second rows of this, also the third and fourth; add the first column to the second; add the first column plus twice the second to the third; and finally add the first column plus three times the second plus three times the third to the fourth, and this becomes

$$\begin{vmatrix} \Theta, & \dot{\Theta}, & \ddot{\Theta}, & \dddot{\Theta}, \\ \Theta', & \dot{\Theta}', & \ddot{\Theta}', & \dddot{\Theta}', \\ \Theta_1, & \dot{\Theta}_1 + \Theta_1, & \ddot{\Theta}_1 + 2\dot{\Theta}_1 + \Theta_1, & \dddot{\Theta}_1 + 3\ddot{\Theta}_1 + 3\dot{\Theta}_1 + \Theta_1 \\ \Theta'_1, & \dot{\Theta}'_1 + \Theta'_1, & \ddot{\Theta}'_1 + 2\dot{\Theta}'_1 + \Theta'_1, & \dddot{\Theta}'_1 + 3\ddot{\Theta}'_1 + 3\dot{\Theta}'_1 + \Theta'_1 \end{vmatrix} = 0$$

In all of these the u has been omitted, simply for convenience in writing.

By decomposing this last into the sum of two determinants and partially expanding the second this may be written in the form

$$\begin{vmatrix} \Theta & \Theta' & \Theta_1 & \Theta'_1 \\ \dot{\Theta} & \dot{\Theta}' & \dot{\Theta}_1 & \dot{\Theta}'_1 \\ \ddot{\Theta} & \ddot{\Theta}' & \ddot{\Theta}_1 & \ddot{\Theta}'_1 \\ \dddot{\Theta} & \dddot{\Theta}' & \dddot{\Theta}_1 & \dddot{\Theta}'_1 \end{vmatrix} = \Theta' \begin{vmatrix} \Theta_1 & \Theta'_1 & \dot{\Theta} \\ 2\dot{\Theta}_1 + \Theta_1 & 2\dot{\Theta}'_1 + \Theta'_1 & \ddot{\Theta} \\ 3\ddot{\Theta}_1 + 3\dot{\Theta}_1 + \Theta_1, & 3\ddot{\Theta}'_1 + 3\dot{\Theta}'_1 + \Theta'_1, & \dddot{\Theta} \end{vmatrix}$$

$$- \Theta \begin{vmatrix} \dot{\Theta}', & \Theta_1, & \Theta'_1 \\ \ddot{\Theta}', & 2\dot{\Theta}_1 + \Theta_1, & 2\dot{\Theta}'_1 + \Theta'_1 \\ \dddot{\Theta}', & 3\ddot{\Theta}_1 + 3\dot{\Theta}_1 + \Theta_1, & 3\ddot{\Theta}'_1 + 3\dot{\Theta}'_1 + 3\dot{\Theta}'_1 + \Theta'_1 \end{vmatrix}$$

The left-hand side of this equation is the Wronskian of Θ, Θ', Θ_1, Θ'_1, or say (using Muir's notation) $W(\Theta, \Theta', \Theta_1, \Theta'_1)$.

By choosing other sets of values of v, w and t, a number of other formulæ may be obtained; it is, however, scarcely worth while writing them down.

I do not know that the following relation has ever been noticed, viz. writing sn, cn, dn, for sn u, cn u, dn u, and denoting by accents the successive differentiations of these functions with respect to u, we have

$$\begin{aligned} \mathrm{sn}' &= \mathrm{cn}\,\mathrm{dn}, & \mathrm{sn}'' &= -\,\mathrm{sn}\,(\mathrm{dn}^2 + k^2\,\mathrm{cn}^2) \\ \mathrm{cn}' &= -\,\mathrm{sn}\,\mathrm{dn}, & \mathrm{cn}'' &= -\,\mathrm{cn}\,(\mathrm{dn}^2 - k^2\,\mathrm{sn}^2) \\ \mathrm{dn}' &= -\,k^2\,\mathrm{sn}\,\mathrm{cn}, & \mathrm{dn}'' &= -\,\mathrm{dn}\,(k^2\,\mathrm{cn}^2 - k^2\,\mathrm{sn}^2) \end{aligned}$$

and substituting these values in the determinant

$$\begin{vmatrix} \mathrm{sn} & \mathrm{sn}' & \mathrm{sn}'' \\ \mathrm{cn} & \mathrm{cn}' & \mathrm{cn}'' \\ \mathrm{dn} & \mathrm{dn}' & \mathrm{dn}'' \end{vmatrix}$$

we find after some very simple reductions which need not be given,

$$\begin{vmatrix} \mathrm{sn} & \mathrm{sn}' & \mathrm{sn}'' \\ \mathrm{cn} & \mathrm{cn}' & \mathrm{cn}'' \\ \mathrm{dn} & \mathrm{dn}' & \mathrm{dn}'' \end{vmatrix} = -k'^2, \text{ or } W(\mathrm{sn}\,u,\ \mathrm{cn}\,u,\ \mathrm{dn}\,u) = -k'^2$$

If instead of sn, cn, dn we write sn^α, cn^α, dn^α, we have

$$\begin{aligned} (\mathrm{sn}^\alpha u)' &= \alpha\, \mathrm{sn}^{\alpha-1} u\, \mathrm{cn}\, u\, \mathrm{dn}\, u, \text{ etc.} \\ (\mathrm{sn}^\alpha u)'' &= \alpha(\alpha-1)\, \mathrm{sn}^{\alpha-2} u\, \mathrm{cn}^2 u\, \mathrm{dn}^2 u - \alpha\, \mathrm{sn}^\alpha u (\mathrm{dn}^2 u + k^2 \mathrm{cn}^2 u) \\ (\mathrm{cn}^\alpha u)'' &= \alpha(\alpha-1)\, \mathrm{cn}^{\alpha-2} u\, \mathrm{dn}^2 u\, \mathrm{sn}^2 u - \alpha\, \mathrm{cn}^\alpha u (\mathrm{dn}^2 u - k^2 \mathrm{sn}^2 u) \\ (\mathrm{dn}^\alpha u)'' &= \alpha(\alpha-1)\, \mathrm{dn}^{\alpha-2} u\, k^4 \mathrm{sn}^2 u\, \mathrm{cn}^2 u - \alpha\, \mathrm{dn}^\alpha u (k^2 \mathrm{cn}^2 u - k^2 \mathrm{sn}^2 u) \end{aligned}$$

and consequently

$$\begin{vmatrix} \mathrm{sn}^\alpha u & (\mathrm{sn}^\alpha u)', & (\mathrm{sn}^\alpha u)'' \\ \mathrm{cn}^\alpha u & (\mathrm{cn}^\alpha u)', & (\mathrm{cn}^\alpha u)'' \\ \mathrm{dn}^\alpha u & (\mathrm{dn}^\alpha u)', & (\mathrm{dn}^\alpha u)'' \end{vmatrix}$$

$$= \alpha^2\, \mathrm{sn}^{\alpha-1} u\, \mathrm{cn}^{\alpha-1} u\, \mathrm{dn}^{\alpha-1} u \begin{vmatrix} \mathrm{sn}^2 u, & 1, & (\alpha-1)\,\mathrm{cn}^2 u, & \mathrm{dn}^2 u - \mathrm{sn}^2 u (\mathrm{dn}^2 u + k^2 \mathrm{cn}^2 u) \\ \mathrm{cn}^2 u, & -1, & (\alpha-1)\,\mathrm{dn}^2 u, & \mathrm{sn}^2 u - \mathrm{cn}^2 u (\mathrm{dn}^2 u - k^2 \mathrm{sn}^2 u) \\ \mathrm{dn}^2 u, & -k^2, & (\alpha-1)\,\mathrm{sn}^2 u, & \mathrm{cn}^2 u - \mathrm{dn}^2 u (k^2 \mathrm{cn}^2 u - k^2 \mathrm{sn}^2 u) \end{vmatrix}$$

Decomposing the determinant on the right into the sum of two this is

$$= \alpha^2(\alpha-1)\, \mathrm{sn}^{\alpha-1} u\, \mathrm{cn}^{\alpha-1} u\, \mathrm{dn}^{\alpha-1} u \begin{vmatrix} \mathrm{sn}^2 u, & 1, & \mathrm{cn}^2 u\, \mathrm{dn}^2 u \\ \mathrm{cn}^2 u, & -1, & \mathrm{dn}^2 u\, \mathrm{sn}^2 u \\ \mathrm{dn}^2 u, & -k^2, & \mathrm{sn}^2 u\, \mathrm{cn}^2 u \end{vmatrix}$$

$$+ \alpha^2 \mathrm{sn}^{\alpha-1} u\, \mathrm{cn}^{\alpha-1} u\, \mathrm{dn}^{\alpha-1} u \begin{vmatrix} \mathrm{sn}\, u, & \mathrm{cn}\, u\, \mathrm{dn}\, u, & -\mathrm{sn}\, u (\mathrm{dn}^2 u + k^2 \mathrm{cn}^2 u) \\ \mathrm{cn}\, u, & -\mathrm{dn}\, u\, \mathrm{sn}\, u, & -\mathrm{cn}\, u (\mathrm{dn}^2 u - k^2 \mathrm{sn}^2 u) \\ \mathrm{dn}\, u, & -k^2 \mathrm{sn}\, u\, \mathrm{cn}\, u, & -\mathrm{dn}\, u (k^2 \mathrm{cn}^2 u - k^2 \mathrm{sn}^2 u) \end{vmatrix}$$

The value of the first of these determinants is easily seen to be k'^2, and of the second $-k'^2$, so we have finally

$$\begin{vmatrix} \mathrm{sn}^\alpha u, & (\mathrm{sn}^\alpha u)', & (\mathrm{sn}^\alpha u)'' \\ \mathrm{cn}^\alpha u, & (\mathrm{cn}^\alpha u)', & (\mathrm{cn}^\alpha u)'' \\ \mathrm{dn}^\alpha u, & (\mathrm{dn}^\alpha u)', & (\mathrm{dn}^\alpha u)'' \end{vmatrix} \text{ or } W(\mathrm{sn}^\alpha u,\ \mathrm{cn}^\alpha u,\ \mathrm{dn}^\alpha u) = \alpha^2(\alpha-2) k'^2\, \mathrm{sn}^{\alpha-1} u\, \mathrm{cn}^{\alpha-1} u\, \mathrm{dn}^{\alpha-1} u$$

For $\alpha = 1$ the right-hand side becomes $= -k'^2$ as found above; for $\alpha = 2$ the determinant is identically equal zero. A more general form is readily found by writing in the above determinant $\mathrm{sn}^\alpha u$, $\mathrm{cn}^\beta u$, $\mathrm{dn}^\gamma u$, instead of $\mathrm{sn}^\alpha u$, $\mathrm{cn}^\alpha u$, $\mathrm{dn}^\alpha u$.